PHYSICAL REVIEW E **67**, 035203(R) (2003)

Phase transition in an ensemble of dissipative solitons of a Turing system

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(Received 8 October 2002; published 24 March 2003)

Numerical study of a two-component reaction-diffusion system where Turing spatial structures are formed has revealed the existence of phase transition between condensed and rarefied phases of dissipative solitons. The effect is due to the dependence of soliton structure on control parameters. The transition manifests a hysteresis, which gives evidence for the possibility of phases to coexist.

DOI: 10.1103/PhysRevE.67.035203 PACS number(s): 82.40.Ck, 05.45.Yv, 64.60.Cn, 89.75.Fb

Reaction-diffusion systems are capable to organize structures where localized domains with high concentration of activator are surrounded by the homogeneous background [1–3]. To refer to such element of a pattern, terms autosoliton [3] or dissipative soliton (DS) [4] have been suggested. In the present paper, these quasiparticle states are referred to as DSs. The DSs are experimentally found in different systems, including electrical networks [5] and chemically reacting media [6,7]. Both stationary and moving DSs are observed in dc-driven quasi-one-dimensional [8] and planar [9,10] semiconductor-gas discharge devices. Schenk *et al.* have theoretically shown [11] that DSs can form stable molecular-like clusters. It is demonstrated in the present work that an ensemble of DSs in a two-dimensional reaction-diffusion system may undergo phase transition between condensed and rarefied states in the course of variation of a control parameter.

This result is obtained by numerical solving of two-component reaction-diffusion equations earlier suggested [9] to interpret formation of patterns in planar semiconductor-gas discharge gap systems driven by dc voltage. While such an approach takes into account some specific features of the gas discharge device, the equations used are a particular case of the general class of two-component reaction-diffusion schemes. They are

$$\frac{\partial U}{\partial t}=\frac{U_0-\gamma\overline{N}-U}{\tau_U}-cNU+D_U\Delta U, \tag{1}$$

$$\frac{\partial N}{\partial t}=-\frac{N}{\tau_N}+NU\left[a+b\left(\frac{N}{N+N^*}\right)^2\right]+D_N\Delta N, \tag{2}$$

where the variables U and N are the voltage drop on the discharge gap and density of charge carriers in the gap, correspondingly. The first equation describes charging the capacity of the gap from a voltage source U_0 with the characteristic time τ_U and its discharging due to free carriers in the gap; c is the constant of the discharging. The presence of $\gamma\overline{N}$ takes into account the global negative feedback that is considered to be proportional to the total number of carriers in the gap. To include it into the local Eq. (1), its value is normalized to the average density

$$\overline{N}=S^{-1}\int_S N dS,$$

where S is the area of the system.

The dynamics of density of carriers is governed by their decay with the lifetime τ_N and the avalanche multiplication, the first and second terms on the right hand side of Eq. (2); a, b, and N^* are constants defining the efficiency of the autocatalytic avalanche process in dynamics of carriers. Last terms in Eqs. (1) and (2) describe the diffusion spreading of variables in the plane of the system. Actually, the equations refer to a two-dimensional space, while changes of variables are considered only in lateral direction. Note that variables N and U may be referred to as activator and inhibitor of the problem of the system's dynamics.

In calculations, we consider the set of parameters in Eqs. (1) and (2) for a model case where the homogeneous state $N>0$ bifurcates to a stationary hexagonal pattern when U_0 reaches some critical value U_0^i. Analogously to those applied in Ref. [9], the main parameters of calculation are $\tau_U=10^{-2}$ sec, $\tau_N=10^{-3}$ sec, $a=1$; $b=0.4$, $N^*=1.5\times10^5$ cm^{-3} $D_U=0.625$ cm^2/sec, $D_N=0.045$ cm^2/sec, and $c=1.64\times10^{-4}$ cm^3/sec;

The instability occurs via the diffusion (Turing) mechanism. The characteristic wave vector of a formed stationary periodic structure is close to that for neutral modes at $U_0=U_0^i$. The bifurcation is a subcritical one, that is, the patterned state exists also in a range of control parameters where the homogeneous solution is linearly stable.

There may be applied in a computing routine two ways of getting a stationary pattern in the domain of subcriticality: (i) starting from a state where patterns have grown spontaneously, to go here via a variation of control parameters; (ii) spatially homogeneous fields of variables are perturbed locally. When the perturbation is removed, the pattern may remain in the system (see, e.g., Ref. [3]). In the present work, the first method of preparing an initial pattern which then, on later stages of computation, is used to study effects of interaction of DSs has been exploited. A relatively low noise of the variable N, whose amplitude is in the range $(10^{-3}$–$10^{-5})N$, has been applied in calculations.

Different number of DSs may exist in the system in the subcriticality domain [3]; the minimum quantity is, evidently, one. A bifurcation diagram for the system, including a one-DS state, is given in Fig. 1 for some set of parameters. Here, amplitudes of activator for a DS and for the homogeneous background are presented as a function of U_0 for the global load $\gamma=0$.

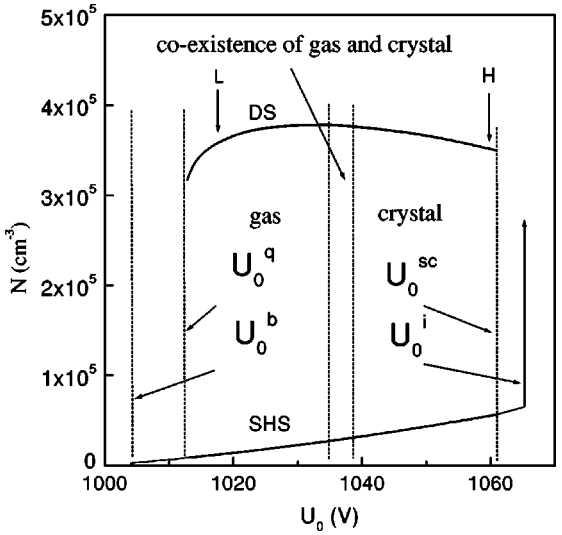

FIG. 1. Part of the bifurcation diagram for Eqs. (1) and (2). Under an increase in feeding voltage, the conductive spatially homogeneous state (SHS) $N>0$ is established at the critical (breakdown) voltage U_0^b. It is linearly unstable against formation of periodic hexagonal structure at $U_0=U_0^i$. A stationary dissipative soliton (DS) of the shown amplitude exists in the range $U_0^q<U_0<U_0^{sc}$ (Fig. 2 shows spatial profiles of DS calculated at L and H points of the diagram). In an ensemble, DSs form either a ''gaseous,'' or ''crystalline'' phase. They can also coexist in some range of varying U_0. The data are obtained at $\gamma=0$ for the square domain of the dimension $L=3.5$ cm. For details of the calculation see the present text.

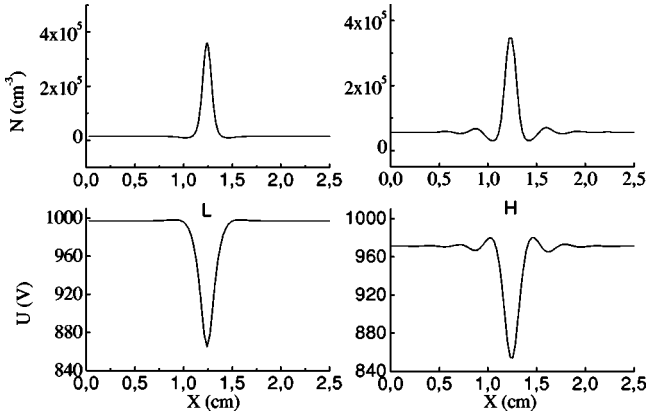

FIG. 2. Calculated spatial profiles of activator N and inhibitor U for DS at low (L) and high (H) values of U_0. Data refer to voltages L and H marked at the bifurcation diagram in Fig. 1.

The range of existence of a one-DS state is confined: At diminishing U_0, it is bounded by a critical voltage U_0^q at which the DS solution disappears. The upper bound on its existence is defined, at the given values of parameters, by the process of the self-completion of the hexagonal periodic pattern on the primary DS as on a seed [9,12]. In the case $\gamma=0$, it then occupies all the available space. Then it can be remarked that the critical value of voltage for such a process $U_0^{sc}<U_0^i$. We also notice that while applying other parameters of Eqs. (1) and (2), the right boundary of the bifurcation diagram for a one-DS state may be determined not by the self-completion process, but by a division of the primary DS [3,13].

The structure of DS varies as U_0 changes, see Fig. 2. It is remarkable that, close to (but below of) U_0^{sc}, the tails in distributions of both activator and inhibitor manifest expressed spatial oscillations. (We notice that this peculiarity has been also observed in experiments made on the cryogenic planar gas discharge system [9,10]). On the contrary, in the vicinity of U_0^q, this feature is not seen.

In the range of stability of a DS, it becomes possible to create a rarefied ensemble of DSs in a spatially extended domain. In relation to the sensitivity of the DS's structure to the magnitude of the control parameter, then an influence of this effect on interaction of the considered quasiparticle objects can be discussed.

In Fig. 3, a sequence of the system's states that is observed at varying U_0 is represented, all other parameters of Eqs. (1) and (2) are kept unchanged. When U_0 is slightly increased above U_0^i, a spontaneously formed hexagon pattern occupies all the available space, Fig. 3(a). The pattern is, essentially, a periodic arrangement of maxima in the spatial distribution of activator. Then decreasing U_0, a state can be reached where the number of maxima starts to decrease at a further decrease in voltage. A stage in the course of this decrease is shown in Fig. 3(b). At the conditions of calculations, it represents not a stationary state; Namely, U_0 has been decreased rather fast. As a result, some activator maxima have died out, whereas the spatial configuration of the system is not a stationary one that would correspond to the current U_0 value. In other words, the rate in U_0 decrease that induces the quenching of some DSs has been higher than that needed for the adiabatic process of the spatial reconstruction of the pattern containing DSs left in the system.

A rarefied structure like that shown in Fig. 3(b) can now be used as an initial pattern for studying effects of interaction of DSs in the following evolution of the system, for a fixed value of the control parameter. In order to exclude an additional influence on such a process of events either of further quenching of DSs, or of their generation via the self-

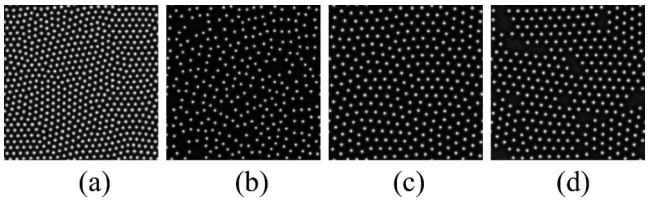

FIG. 3. Spatial distributions of activator N obtained in the course of the following stages of numerical solving of Eqs. (1) and (2): (a) hexagonal stationary pattern spontaneously forms when voltage U_0 exceeds U_0^i; (b) some maxima in distribution of N are quenched as U_0 is diminished to low enough values inside the subcriticality domain. There is observed a phase transition from the ''gaseous'' state of DSs with the repulsive interaction between them (c), to the condensed phase of DSs (d) where attractive forces dominate their behavior, as U_0 increases. The number of DSs remains constant in the sequence of parts (b)–(d). Calculations are done for the square domain of dimension $L=7$ cm.

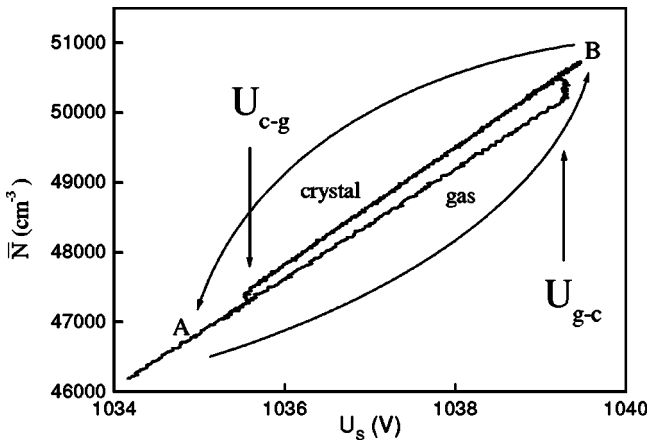

FIG. 4. Average density of activator $\bar{N}$ versus voltage drop U_s on the structure in the domain of transition from purely ''gaseous'' state *A* to the ''crystalline'' one *B*. The effective conductance of the system is higher for the condensed phase of DSs as compared to the gaseous phase, upper, and lower branches of the hysteresis loop, correspondingly. To make the computing time reasonably short, the sequence of quasiadiabatical states is obtained for a rather dense ensemble of DSs, their number over the calculated area of $L=3.5$ cm being 78. The feeding voltage U_0 is changing in the range 1265–1300 V at $\gamma=5\times10^{-3}$.

completion process, this study is performed in range of the control parameter where the number of DSs inside the system's area remains constant in the course of its relaxation. In Fig. 3(c), the asymptotic spatial configuration of the system near the left boundary of existence of a stable solitary DS is shown, while Fig. 3(d) represents a stage in the system's evolution when U_0 is close to the critical value for the self-completion process.

In both cases, spatially ordered phases of DSs are observed. For low U_0, the ordering of the structure is due to repulsive interaction of DSs, so that their ensemble occupies all the available space. On the contrary, for high U_0, the ordered state occupies only a part of the system which gives evidence for condensation of DSs into a compact phase. Hence, the transition between rarefied (gaseous) [14] and compact (crystalline) phases may be induced via varying the control parameter. We point out that number of DSs left in the system at $U_0^q<U_0<U_0^{sc}$ is determined both by U_0 value and by the system's prehistory. In such a way, at the repulsive interaction of DSs one can get (in a studied case of a low noise) a multiplicity of spatially ordered phases with different lattice spacings. On the contrary, for high values of $U_0(U_0<U_0^{sc})$, number of particles in the condensed phase may be different, while the lattice period is nearly fixed.

The phase transition influences transport properties of the system which can be characterized by its conductance $\sigma\sim\bar{N}/U_s$, where $U_s=U_0-\gamma\bar{N}$. Figure 4 shows an example of variation of $\bar{N}$ at a slow (quasiadiabatic) change in the feeding voltage. It has been observed that transitions from gaseous state *A* to the crystalline one *B* and back are specified by the *S*-type behavior of the transport curve. (In order to reveal this effect, an external load $\gamma>0$ has been applied in calculations). This indicates the presence of the negative differential conductance of the system in the course of a transition. An expressed hysteresis in the data gives evidence for the possibility of a coexistence of crystalline and gaseous phases of DSs in some range of changing the control parameter.

We also notice that, near the point of the phase transition, the system exemplifies the slowing down of dynamics of transient processes. The characteristic form of relaxation is not, however, the exponential one: The speed of relaxation depends on how distant the system is from a final steady state. For initial phases of the process, the rate of relaxation is rather high. In case of the transition ''gas → crystal,'' this stage corresponds to a condensation of nearby DSs into quasimolecular aggregates (stable spatial configurations for some of these aggregates have been treated in Ref. [11]). Then more slow processes of aggregation proceed, which are related to the inclusion of more distant DSs into their condensation, as well as to the drift and slow turning of primary formed clusters of DSs. The overall process is characterized by a spectrum of relaxation times (by the stretched kinetics).

It is appropriate to stress that Eqs. (1) and (2) are suggested to describe pattern formation in an electrical system [9]. There the DS is actually a filament of electrical current. The present consideration suggests that the spatial configuration of a multifilament pattern of current can be controlled with feeding voltage. The pattern may be either in the form of a spatially dispersed ensemble of filaments (which occurs at the state where the repulsive interaction of filaments takes place), or in the form of a compact bunch of them. In the latter case, we have the bounded-in-space current channel that has a spatially modulated internal structure according to the Turing mechanism.

Processes similar to that considered in the present study are expected to exist for other models including those being suggested to describe self-organization of chemical media. Indeed, Eqs. (1) and (2) are, essentially, a variant of the Gray-Scott model [15] introduced for the description of nonlinear chemical reactors. We also remark that phase transition from a dispersed to a condensed state of DSs is, as a matter of fact, a bifurcation occurring in the nonlinear system when its steady state solution changes at some (critical) value of a control parameter.

While DS is a generic element of patterns in reaction-diffusion media, considered phase transitions can be expected to exist in various experimental systems. We believe, however, that the semiconductor-gas discharge device might occur as a good candidate for searching these effects. It is a flexible experimental system with two natural control parameters, which are the feeding voltage and the intensity of light which controls the conductivity of the semiconductor component. In experiments on these systems, both compact hexagonal clusters and patterns that consist of multiple spots of the gas discharge plasma have been observed [9,10]. However, up to now, the existence of a transition between two distinct phases that would be specified by a hysteretical behavior similar to that shown in Fig. 4 has not been proved.

One has to point out, however, that, in a pure case, the effect manifests itself as a reconstruction of the (quasi)stationary pattern when control parameters are changed uni-

formly over the active area, whereas number of DSs is constant. This requires the high homogeneity of an experimental system so that regularities of interaction of DSs in different parts of the system are the same.

Finally, we remark that a ''free'' DS that may coexist with the condensed phase seems to be more sensitive to the influence of natural noise of a real system as compared to a cluster of particles. Noise gives rise to an appearance of local transversal gradients in an active medium, which may stimulate the propagation of a DS [16]. In such a case, the whole pattern may have the appearance of a stationary cluster of particles, which is immersed into an ensemble of stochastically traveling DSs.

The author acknowledges support of the work by the Russian Foundation for Basic Research, Grant No. 00-15-96750.

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